

CLIFFORD HOPF-GEBRA AND BI-UNIVERSAL HOPF-GEBRA

ZBIGNIEW OZIEWICZ

ABSTRACT. For a scalar product ξ on co-vectors, the Clifford co-product Δ^ξ of multivectors is calculated from the dual Clifford algebra. With respect to this co-product Δ^ξ , unit is not group-like and vectors are not primitive. For a scalar product η on vectors the Clifford product \wedge^η and the Clifford co-product Δ^ξ fits to the bi-gebra with respect to the family of the (pre)-braids. The Clifford bi-gebra is in a braided category iff $\xi = 0$ or $\eta = 0$.

CONTENTS

1. Multi-ary Bi-gebra	1
2. Clifford Co-gebra and Antipode	3
3. Co-gebra (Co-field) of Co-complex Numbers and Antipode	5
4. Hopf-gebra and Bi-gebra of Complex Numbers	5
5. Bi-universal Hopf-gebra	6
References	8

1. MULTI-ARY BI-GEBRA

A multiplicative category is a category with a bifunctor of bin-ary operation, an annihilation $2 \rightarrow 1$, denoted by two initial leaves and one node. A co-multiplicative category possess a binary co-operation, a creation process $1 \rightarrow 2$. A pairing $2 \rightarrow 0$, a bin-ary annihilation $2 \rightarrow 1$, a bin-ary creation $1 \rightarrow 2$, and bin-ary scattering $2 \rightarrow 2$ are represented by the prime graph nodes in Diagram 1. All diagrams are directed and is recommendent to read them from the top to the bottom.



DIAGRAM 1. A pairing, binary multiplication (annihilation),
binary co-product *i.e.* creation and a scattering (braid)

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Throughout this paper \mathbb{k} denotes a commutative ring. A pair of \mathbb{k} -modules, say A and B , with a pairing $A \otimes B \rightarrow \mathbb{k}$, is said to be a dual pair. Dual pair of al-gebras (co-gebras) extends to pair of co-gebras (al-gebras) and these structures, al-gebra and co-gebra, may close to bi-gebra for a family of pre-braids. We calculate these structures from the assumptions displayed on Diagrams 2-4.



DIAGRAM 2. The product - co-product duality

The Diagram 2 imply that whenever product (co-product) is in variety then co-product (product) is in 'the same' co-variety. Diagram 2 extends to duality between n -ary multiplication and n -ary co-multiplication as shown for tern-ary operations, $1 \leftrightarrow 3$, on Diagram 3.

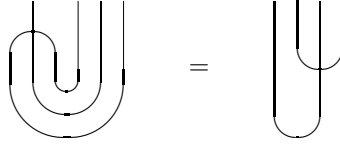


DIAGRAM 3. The tern-ary co-product - product duality

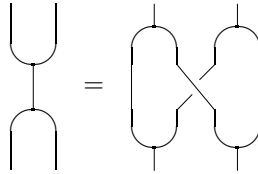


DIAGRAM 4. The bin-ary bi-gebra with one (pre)-braid

A tern-ary bi-gebra is defined on Diagram 5 and this made clear the definition of the multi-ary bigebra.

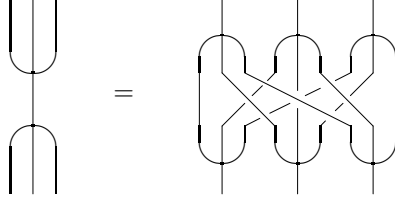


DIAGRAM 5. A tern-ary bi-gebra with nine scatterings (braids)

2. CLIFFORD CO-GEBRA AND ANTIPODE

Dirac in 1928 predicted the existence of an anti-matter, spin $\frac{1}{2}$ positrons, in terms of the Clifford algebra. To understand an anti-matter for any spin we need an action of the Clifford algebra \mathcal{Cl} in a tensor product of \mathcal{Cl} -modules. The differential Dirac operator for mesons of zero and higher spins needs an action of the Clifford algebra on a tensor product of Clifford algebras and this action is illustrated on Diagram 6. This action depends on choosen co-product, the simplest one is known as the Duffin & Kemmer & Petiau co-product [Duffin 1938, Kemmer 1939, 1943]. The main problem is the classification of the co-products which fit the Clifford algebra into bi-gebra. The present paper is the introduction into this subject.

A category is said to be autonomous if $\forall M \in \text{obj}, \exists !$ a left dual M^* and a right dual *M [Freyd & Yetter 1992]. An autonomous category is said to be pivotal if $M^* \simeq {}^*M$.

Let M be \mathbb{k} -module and η and ξ be scalar products,

$$\begin{aligned}\eta &\in \text{lin}(M, M^*) \simeq M^{*\otimes 2} \simeq \text{lin}(M^{\otimes 2}, \mathbb{k}), \\ \xi &\in \text{lin}(M^*, M) \simeq M^{\otimes 2} \simeq \text{lin}(M^{*\otimes 2}, \mathbb{k}).\end{aligned}$$

A pair of the mutually dual Clifford \mathbb{k} -algebras is paired by determinant (scalar product independent),

$$(2.1) \quad \mathcal{Cl}(M, \eta) \simeq \{M^\wedge, \wedge^\eta\}, \quad \mathcal{Cl}(M^*, \xi) \simeq \{M^{*\wedge}, \wedge^\xi\},$$

$$(2.2) \quad M^{*\wedge} \otimes M^\wedge \xrightarrow{\det} \mathbb{k}.$$

A ξ -dependent co-multiplication $\Delta^\xi : M^\wedge \longrightarrow M^\wedge \otimes M^\wedge$ is calculated from the product - co-product duality of Diagram 2.

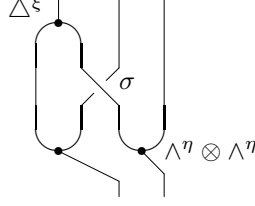


DIAGRAM 6. A co-product dependent action of $\mathcal{C}\ell$ on tensor product of $\mathcal{C}\ell$ -modules

In the sequel $\{e_\mu \in M\}$ denotes a basis and $\{\varepsilon^\mu \in M^* \equiv \text{lin}(M, \mathbb{k})\}$ is a dual basis, $\varepsilon^\mu e_\nu = \delta_\nu^\mu$. For $1 \in \mathbb{k} < M^\wedge$ and $v, w \in M$,

$$\begin{aligned} \Delta^\xi 1 &= 1 \otimes 1 + \sum \xi(\varepsilon^\mu \otimes \varepsilon^\nu) e_\mu \otimes e_\nu \\ &\quad - \sum \xi(\varepsilon^{\mu_1} \wedge \varepsilon^{\mu_2}, \varepsilon^{\nu_1} \wedge \varepsilon^{\nu_2}) (e_{\mu_1} \wedge e_{\mu_2}) \otimes (e_{\nu_1} \wedge e_{\nu_2}) \\ &\quad - \sum \xi(\text{tri-co-vectors}^{\otimes 2}) \text{trivectors}^{\otimes 2} + \dots + (-)^{\lfloor \frac{1}{2} \text{grade} \rfloor} \dots \end{aligned}$$

$$\Delta^\xi v = 1 \otimes v + v \otimes 1 + \sum \xi(\varepsilon^\mu \otimes \varepsilon^\nu) [e_\mu \otimes (v \wedge e_\nu) - (v \wedge e_\nu) \otimes e_\mu] + \dots$$

$$\begin{aligned} \Delta^\xi(v \wedge w) &= 1 \otimes (v \wedge w) + (v \wedge w) \otimes 1 - v \otimes w + w \otimes v \\ &\quad + \sum \xi(\varepsilon^\mu \otimes \varepsilon^\nu) [(w \wedge e_\mu) \otimes (e_\nu \wedge v) - (v \wedge e_\mu) \otimes (e_\nu \wedge w)] + \dots \end{aligned}$$

If $\xi = 0$ then $1 \in \mathcal{C}\ell$ is group-like, vectors are $(1, 1)$ -primitive and $\Delta^{\xi=0}$ is the Duffin & Kemmer & Petiau co-product [Duffin 1938, Kemmer 1939, 1943].

The above co-product Δ^ξ (as well as (3.1) late on) is co-unital

$$(2.3) \quad \varepsilon \in \text{lin}(M^\wedge, \mathbb{k}), \quad \mathbb{k} \ni \varepsilon w = \begin{cases} 0 & \text{if grade } w \neq 0, \\ w & \text{if grade } w = 0. \end{cases}$$

However co-unit (2.3) is not an algebra map iff $\eta \neq 0$ and unit $u \in \text{lin}(\mathbb{k}, M^\wedge)$ is not cogebra map iff $\xi \neq 0$,

$$\begin{aligned} \eta \neq 0 &\iff \varepsilon \notin \text{alg}(\wedge^\eta, \mathbb{k}), \\ \xi \neq 0 &\iff u \notin \text{cog}(\mathbb{k}, \Delta^\xi). \end{aligned}$$

Conjecture 2.1. A condition $\xi \circ \eta \neq \text{id}_M$ is a necessary and sufficient condition that exists an antipode $S \in \text{End}(M^\wedge)$ (see example below).

If $\xi = 0$ then antipode exists and $S|M^{\wedge 2} = 0$.

3. CO-GEBRA (CO-FIELD) OF CO-COMPLEX NUMBERS AND ANTIPODE

In the sequel if $\alpha \in M^*$ then $\alpha^2 \in \mathbb{k}$ stands for $\xi(\alpha \otimes \alpha)$ and if $v \in M$ then $v^2 \in \mathbb{k}$ stands for $\eta(v \otimes v)$.

In sections 3-4 $\dim_{\mathbb{R}} M = 1$. Let $i \in M$. If $\eta(i \otimes i) = -1$ then $\mathbb{C} \simeq \mathcal{C}\ell(M, \eta)$.

If $\xi \circ \eta = \text{id}_M$ and $\alpha v = 1 \in \mathbb{k}$, then $\alpha^2 v^2 \equiv \xi(\alpha \otimes \alpha) \eta(v \otimes v) = 1$.

For $j \in \mathbb{C}^* \equiv \text{lin}_{\mathbb{R}}(\mathbb{C}, \mathbb{R})$,

$$\begin{aligned}
 \wedge^\eta(1 \otimes 1) &= 1, \quad \wedge^\eta(i \otimes i) = \wedge_i^\eta i = \eta(i \otimes i) \in \mathbb{R}, \\
 \wedge^\eta(1 \otimes i) &= \wedge_1^\eta i = i, \quad \wedge^\eta(i \otimes 1) = \wedge_i^\eta 1 = i, \\
 \Delta^\xi 1 &= 1 \otimes 1 + \xi(j \otimes j) i \otimes i, \\
 \Delta^\xi i &= 1 \otimes i + i \otimes 1.
 \end{aligned}
 \tag{3.1}$$

For $z \in \mathbb{C}$ and $i^2 j^2 = 1$, $\Delta z = 1 \otimes z + \frac{1}{i^2} i \otimes iz$. From now on

$$a \equiv i^2 j^2 = \eta(i \otimes i) \xi(j \otimes j) \in \mathbb{R}.$$

Proposition 3.1. *With respect to co-unit (2.3), an antipode $S \in \text{lin}(\mathbb{C}, \mathbb{C})$ exists iff $\xi \circ \eta \neq \text{id}_M$, i.e. iff $a \neq 1$, and then*

$$S 1 = \frac{1}{1-a}, \quad S i = -\frac{i}{1-a}. \tag{3.2}$$

4. HOPF-GEBRA AND BI-GEBRA OF COMPLEX NUMBERS

Diagram 4 is a relation among three tensors, a product \wedge , co-product Δ and a bin-ary scattering σ . The purpose of this section is to determine from Diagram 4 the set of all possible scatterings $\sigma \in \text{End}(\mathbb{C}^{\otimes 2})$ for (η, ξ) -dependent product and coproduct (3.1) on two-dimensional \mathbb{R} -space span by $\{1, i\} \in \mathbb{C}$. Then the set $\{\wedge^\eta, \Delta^\xi, \sigma\}$ is a bin-ary bi-gebra.

If $i^2 j^2 \neq 1$ then exists the unique scattering $\sigma \in \text{End}_{\mathbb{R}}(\mathbb{C}^{\otimes 2})$,

$$\begin{aligned}
 \sigma(1 \otimes 1) &= \left(1 - \frac{a^2}{1-a}\right) 1 \otimes 1 - \frac{j^2}{1-a} i \otimes i, \\
 \sigma(i \otimes i) &= -\frac{1}{1-a} (i \otimes i + i^2 \cdot 1 \otimes 1), \\
 \sigma(1 \otimes i) &= \frac{1}{1-a} (i \otimes 1 + a \cdot 1 \otimes i), \\
 \sigma(i \otimes 1) &= \frac{1}{1-a} (1 \otimes i + a \cdot i \otimes 1).
 \end{aligned}
 \tag{4.1}$$

The minimum polynomial of (4.1) is of the fourth order

$$b \equiv \frac{1+i^2 j^2}{1-i^2 j^2}, \quad (\sigma + \text{id}) \circ (\sigma - b \cdot \text{id}) \circ (\sigma^2 + ab \cdot \sigma - b \cdot \text{id}) = 0.$$

Therefore σ (4.1) is invertible iff $i^2 j^2 \neq \pm 1$. We conjecture that $\sigma \in \text{End}(M^{\otimes 2})$ (4.1) is a (pre)-braid operator i.e. σ is a solution of the Artin braid equation (this

indeed is the case if $i^2 j^2 = 0$),

$$(4.2) \quad (\sigma \otimes \text{id}_M) \circ (\text{id}_M \otimes \sigma) \circ (\sigma \otimes \text{id}_M) = (\text{id}_M \otimes \sigma) \circ (\sigma \otimes \text{id}_M) \circ (\text{id}_M \otimes \sigma),$$



Proposition 4.1. *The Clifford bi-gebra $\{\mathbb{R}^2, \wedge^\eta, \triangle^\xi, \sigma\}$, (3.1)-(4.1), is σ -braided iff $\eta = 0$ or $\xi = 0$.*

The formulas (3.1)-(3.2)-(4.1) describe two-parameter $\{i^2, j^2\}$ -family of Hopf-gebras (and this include the field of complex numbers) for which neither unit nor co-unit are respecting co-product and product respectively.

If $\xi \circ \eta = \text{id}_{\mathbb{C}}$ i.e. if $i^2 j^2 = 1$, then exists 12-parameters family of mappings $\sigma \in \text{End}_{\mathbb{R}}(\mathbb{C}^{\otimes 2})$ which fit to bi-gebra. Among other this include the following solution for $p + q + r = 0 \in \mathbb{R}$,

$$\begin{aligned} \sigma(1 \otimes 1) &= 1 \otimes 1, \\ \sigma(1 \otimes i) &= i \otimes 1 + p \cdot 1 \otimes i, \\ \sigma(i \otimes 1) &= 1 \otimes i + q \cdot i \otimes 1, \\ \sigma(i \otimes i) &= r \cdot i \otimes i - i^2 \cdot 1 \otimes 1. \end{aligned}$$

5. BI-UNIVERSAL HOPF-GEBRA

Some notation	
\mathbb{k}	is a commutative ring
$\mathbb{k}\text{-mod}$	a category of \mathbb{k} -modules (of \mathbb{k} -alphabets)
$\mathbb{k}\text{-alg}$	a category of associative unital \mathbb{k} -algebras
$T : \mathbb{k}\text{-mod} \rightarrow \mathbb{k}\text{-alg}$	the tensor algebra functor
$F : \mathbb{k}\text{-alg} \rightarrow \mathbb{k}\text{-mod}$	the forgetful functor;
\otimes	bifunctor of tensor product: $\mathbb{k}\text{-mod} \times \mathbb{k}\text{-mod} \rightarrow \mathbb{k}\text{-mod}$.
\otimes	means $\otimes_{\mathbb{k}}$ if not otherwise stated;
$\text{lin} \equiv \text{lin}_{\mathbb{k}}, \text{End} \equiv \text{End}_{\mathbb{k}}$	are both sided \mathbb{k} -linear bifunctors;
$M \in \mathbb{k}\text{-mod}$	is a \mathbb{k} -module (a \mathbb{k} -alphabet);
$M^{\otimes} = FTM$	a \mathbb{Z} -graded \mathbb{k} -vocabulary in a \mathbb{k} -alphabet M ;
$M^* \equiv \text{lin}(M, \mathbb{k})$	a dual \mathbb{k} -module of co-vectors.

A bi-associative (i.e. an associative and co-associative) and bi-unital (i.e. unital and co-unital) Hopf-gebra in a braided monoidal category (\equiv a braided Hopf-gebra or a ‘braided group’) has been introduced by Majid in series of papers in years

1991-1993. In [Oziewicz et al. 1995] we generalized a braided Hopf-gebra to pre-braided Hopf-gebra when a braid needs not to be invertible. This generalization was motivated by the following problem: does exist pre-braid for which exists a pre-braided bi-universal (*i.e.* universal and co-universal) Hopf-gebra? This is illustrated by Diagram 4 in the case of the fixed universal product and of the co-universal co-product. We showed that pre-braided bi-universal bi-associative and bi-unital Hopf-gebra exists for zero pre-braid only [Oziewicz et al. 1995].

For a \mathbb{k} -module M , M^\otimes denotes \mathbb{Z} -graded \mathbb{k} -module (not an algebra) *i.e.* a totality of all finite sentences in M .

By definition functors T and F are adjoint [Kan 1958]: bifunctors $\text{lin}_{\mathbb{k}}(\cdot, F\cdot)$ and $\text{alg}_{\mathbb{k}}(T\cdot, \cdot)$ are naturally equivalent. This means that a natural set bijection holds,

$$\begin{aligned}
 (5.1) \quad & \forall M \times A \in \mathbb{k}\text{-mod} \times \mathbb{k}\text{-alg}, \\
 & \text{lin}_{\mathbb{k}}(M, FA) \ni \ell \longleftrightarrow \ell^A \in \text{alg}_{\mathbb{k}}(TM, A), \quad \ell^A|_M \equiv \ell. \\
 & \ell^m \equiv \ell^A = m^\otimes \circ \ell^\otimes \in \text{alg}(TM, A), \\
 & \ell^\Delta \equiv \ell^C = \ell^\otimes \circ \Delta^\otimes \in \text{cog}(C, ShM),
 \end{aligned}$$

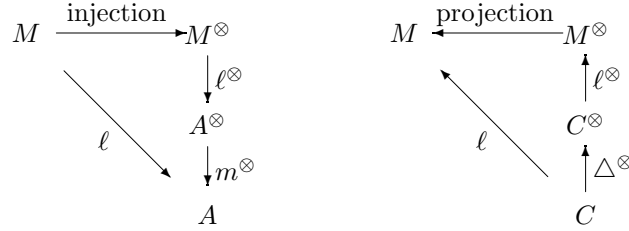


DIAGRAM 7. The universal and co-universal lifts

An example of realization of M -universal tensor \mathbb{k} -algebra is $TM \simeq \{M^\otimes, \otimes\}$ *i.e.* a \mathbb{Z} -graded \mathbb{k} -module M^\otimes of all finite words in an alphabet M with a concatenation \otimes as a multiplication of grade $\otimes = 0$.

An example of realization of M -co-universal co-gebra is $ShM \simeq \{M^\otimes, \text{sh}\}$, sh is short for 'shuffle' [Sweedler 1969], $T(M^*) \simeq (ShM)^*$.

A deformation of bi-gebra is said to be *preferred* if either product or co-product is not deformed [Gerstenhaber & Schack 1992, Bonneau et al. 1994]. One can consider two pre-braid dependent bi-associative preferred deformations of 0-braided bi-universal Hopf-gebra: an universal Hopf-gebra which is not co-universal (an universal product is not deformed) and co-universal Hopf-gebra which is not universal (a co-universal shuffle co-product is not deformed). These families generalize for arbitrary pre-braid the Sweedler construction for the switch [Sweedler 1969, chapter XII].

There exists an unique pre-braid dependent homomorphism (extending an identity mapping on generating space) of universal Hopf-gebra into co-universal Hopf-gebra [Oziewicz et al. 1995, Rózański 1996]. This Hopf-gebra homomorphism

1. is a pre-braid dependent deformation of an identity,
2. commutes with an antipod and
3. for invertible braid coincide with the braid-dependent ‘(anti)symmetrizer’ introduced by Woronowicz in 1989.

The image of this Hopf-gebra homomorphism is said to be an exterior Hopf-gebra. An exterior Hopf-gebra is co-universal and pre-braided.

An open problem is to find necessary and sufficient conditions (on braid, scalar product, Lie algebra, etc.) that exists pre-braided filtered algebra as tensor-dependent quantizations and (the Chevalley) deformations of exterior Hopf-gebra.

REFERENCES

- [1] Bonneau P., M. Flato, Murray Gerstenhaber and G. Pinczon, *The hidden group structure of quantum groups: strong duality, rigidity and preferred deformations* Communications in Mathematical Physics **161** (1994) 125–156
- [2] Duffin R. J., *On the characteristic matrices of covariant system*, Phys. Rev. **54** (1938) 1114
- [3] Freyd P. J. & David N. Yetter, *Coherence theorems via knot theory*, Journal of Pure and Applied Algebra **78** (1992) 49–76 [MR93d:18013]
- [4] Gerstenhaber Murray, Anthony Giaquinto and Samuel D. Schack, *Quantum groups, cohomology and preferred deformations*, in “XXth International Conference on Differential Geometric Methods in Theoretical Physics”, edited by Sultan Catto & Alvany Rocha, World Scientific, Singapore, New Jersey 1992, pp. 529–538
- [5] Gerstenhaber Murray & Samuel D. Schack, *Algebras, bialgebras, quantum groups, and algebraic deformations*, Contemporary Mathematics **134** (1992) 51–92
- [6] Kemmer N., *The particle aspect of meson theory*, Proc. Roy. Soc. **A 173** (1939) 91
- [7] Majid Shahn, *Braided groups and algebraic quantum field theories*, Letters in Mathematical Physics **22** (1991) 167–176
- [8] Majid Shahn, *Braided groups*, J. Pure and Applied Algebra **86** (1993) 187–221
- [9] Majid Shahn, *Transmutation theory and rank for quantum braided groups*, Math. Proc. Camb. Phil. Soc. **113** (1993) 45–70
- [10] Majid Shahn, *Free braided differential calculus, braided binomial theorem and the braided exponential map*, Journal of Math. Phys. **34** (1993) 4843–4856
- [11] Majid Shahn, “Foundations of Quantum Group Theory”, Cambridge University Press 1995
- [12] Oziewicz Z., Paal E. and Rózański J., *Derivations in braided geometry*, Acta Physica Polonica **B 26** (7) (1995) 1253–1273
- [13] Rózański Jerzy, *Braided antisymmetrizer as bialgebra homomorphism*, Reports in Mathematical Physics **38** (2) (1996) 273–277
- [14] Sweedler Moss E., “Hopf algebras”, Benjamin Inc., New York 1969
- [15] Woronowicz Stanisław Lech, *Differential calculus on compact matrix pseudogroups (quantum groups)*, Commun. Math. Phys. **122** (1989) 125–170

UNIVERSITY OF WROCLAW, INSTITUTE OF THEORETICAL PHYSICS, PLAC MAKSA BORNA 9,
50204 WROCLAW, POLAND

Current address: Universidad Nacional Autónoma de México, Facultad de Estudios Superiores,
C.P. 54700 Cuautitlán Izcalli, Apartado Postal # 25, Estado de México

E-mail address: oziewicz@servidor.unam.mx, oziewicz@ift.uni.wroc.pl